Analysis I Lecture 9

Last: Defined sequences

(x_n) is a string of real numbers

(function)

Formally is a map $\mathbb{N} \to \mathbb{R}$

$$\begin{array}{ccc} \mathbb{N} & \to & \mathbb{R} \\ \Psi & & \ni \\ n & \longmapsto & x_n \end{array}$$

<u>Defined</u> - Monotonicity

- Boundedness

Convergence of sequences.

Definition. Let $(x_n)_{n \geq n_0}$ be a sequence

a number $\ell \in \mathbb{R}$ is called

a limit of (x_n) $\left(\lim_{n \to \infty} x_n = \ell \right)$

if $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t.

Notation

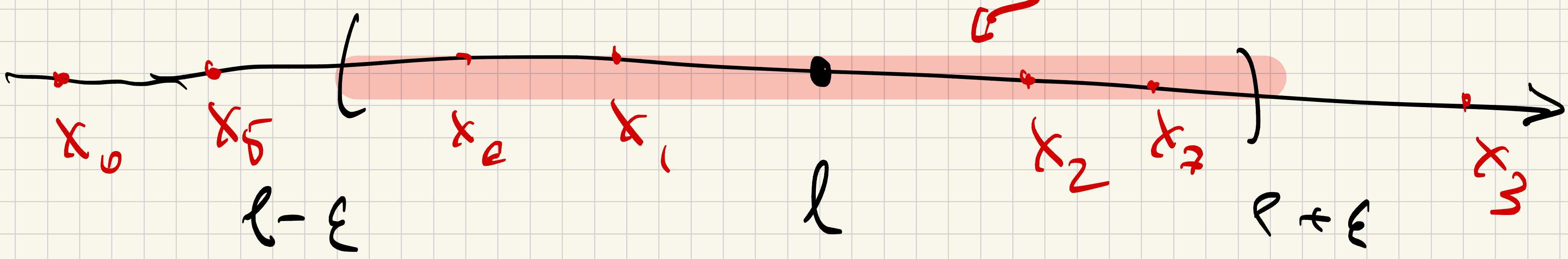
$\forall n > N$ $|x_n - \ell| < \varepsilon$.

If (x_n) has a limit we call it convergient.

$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N}$ s.t.

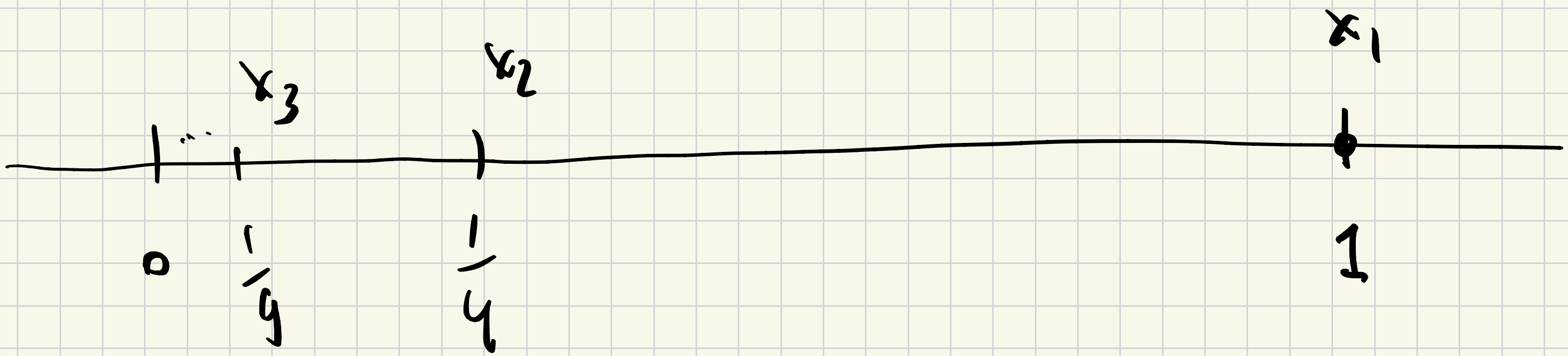
$\forall n > N \quad |x_n - \ell| < \varepsilon.$

Starting from some $N+1$ all x_n live here



Examples • Let $x_n = \frac{1}{n^2}$ for $n \geq 1$

Guess for $\lim_{n \to \infty} \frac{1}{n^2}$ is 0



Let's prove that $\lim_{n\to\infty} \frac{1}{n^2} = 0$

Fix some $\varepsilon > 0$

then we want that

$$\frac{1}{n^2} < \varepsilon \quad \text{for} \quad n > N$$

$$\Leftrightarrow \quad n^2 > \frac{1}{\varepsilon} \quad \Leftrightarrow \quad n > \sqrt{\frac{1}{\varepsilon}}$$

Preperation

$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N}$ s.t.

$\forall n > N \quad |x_n - \ell| < \varepsilon$.

$$\left|\frac{1}{n^2}\right| < \varepsilon$$

since $\ell = 0$

and $x_n = \frac{1}{n^2}$

<u>Proof</u> Indeed for any $\varepsilon > 0$ we can take

$$N = \left\lceil \sqrt{\frac{1}{\varepsilon}} \right\rceil$$

Proof Indeed for any $\varepsilon > 0$ we can take

$$N \geq \left\lceil \sqrt{\frac{1}{\varepsilon}} \right\rceil .$$

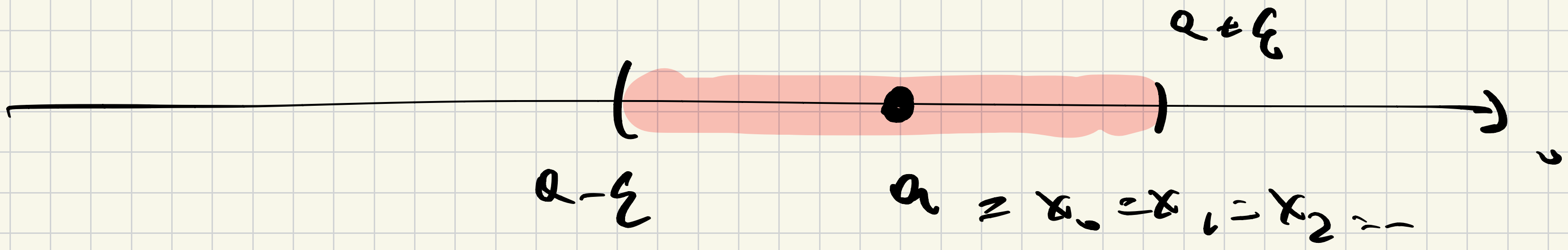
Then $\forall \; n > N$ we will have:

$$n > \left\lceil \sqrt{\frac{1}{\varepsilon}} \right\rceil \geq \sqrt{\frac{1}{\varepsilon}} \iff \left| \frac{1}{n^2} \right| < \varepsilon .$$

Since this worked for any $\varepsilon > 0$ this shows that $\lim\limits_{n \to \infty} \frac{1}{n^2} = 0$

[redacted]

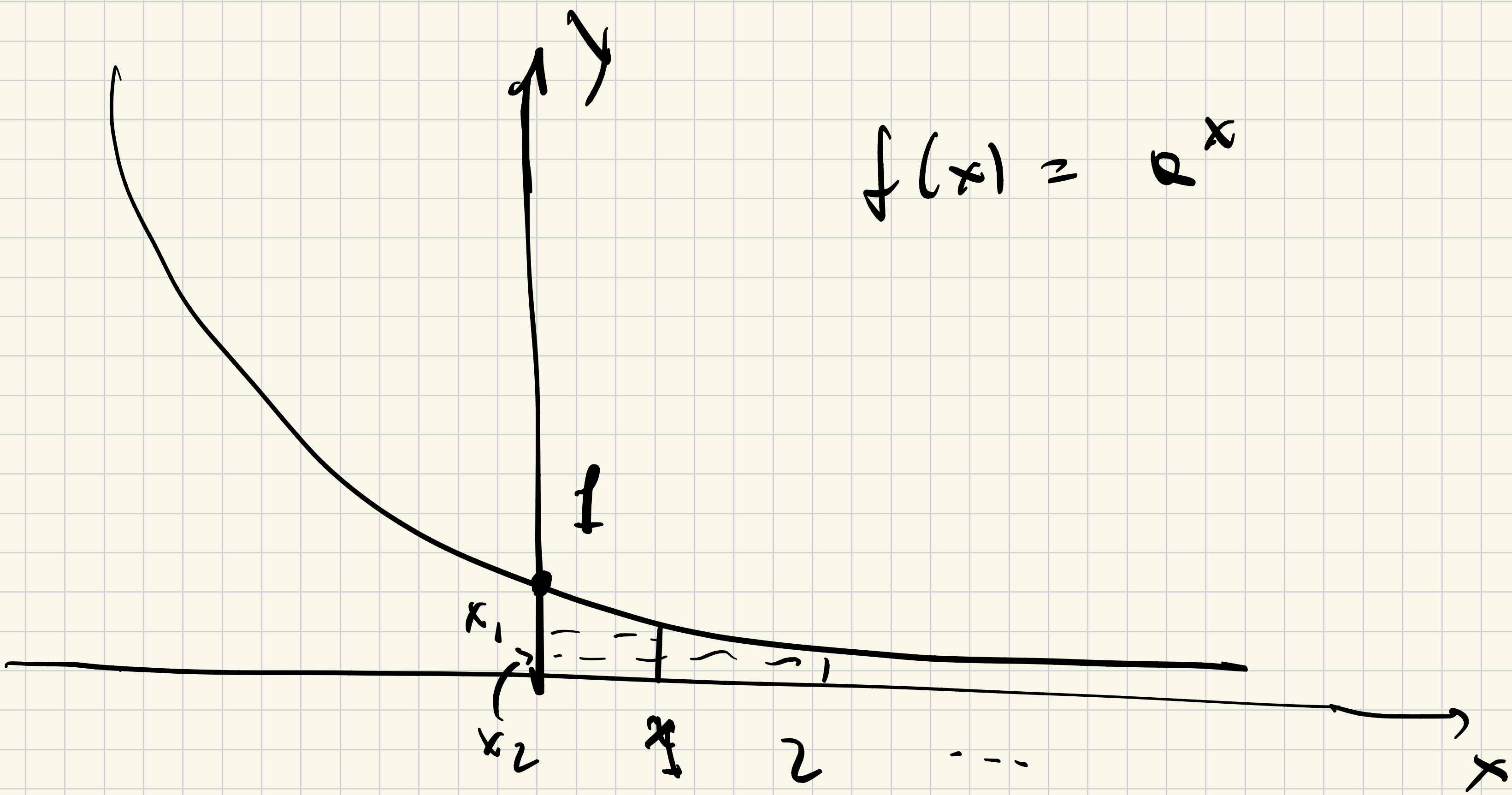
2) Let $a \in \mathbb{R}$ and $x_n = a$ $\forall n \geq 0$



$$\lim_{n \to \infty} x_n = a$$

<u>Proof:</u> For any $\varepsilon > 0$ we can take $N = 0$. Indeed since $x_n = a$ $\forall n$ $|a - x_n| = 0 < \varepsilon$ for all n. ∎

3) Let $0 < a < 1$ and let $x_n = a^n$



A natural guess: $\lim_{n\to\infty} x_n = 0$.

Preparation for the proof:

$\forall \varepsilon \quad \exists N: \quad$ if $n > N$ we have

$$|x_n - 0| < \varepsilon \;\Rightarrow\; a^n < \varepsilon$$

$$|a^n| \qquad e^{\log(a)\cdot n} < e^{\log \varepsilon}$$

$$a^n \qquad e^{\log(a)\cdot n - \log \varepsilon} < 1$$

$$\Rightarrow \log(a)\cdot n - \log \varepsilon < 0$$

$$\log(a)\cdot n - \log\varepsilon < 0 \iff \log(a)\cdot n < \log(\varepsilon)$$

Remember $0 < a < 1 \Rightarrow \log(a) < 0$

So we get that

$$n > \frac{\log(\varepsilon)}{\log(a)}$$

We get that $\boxed{a^n < \varepsilon \text{ if } n > \frac{\log(\varepsilon)}{\log(a)}}$

$$\lim_{n\to\infty} a^n = 0$$

Proof: For any $\varepsilon > 0$ we can take $N = \max\left(\left\lceil \frac{\log(\varepsilon)}{\log(a)} \right\rceil, 0\right)$

then $\forall\, n \geq N$ $\quad n \geq \frac{\log(\varepsilon)}{\log(a)}$

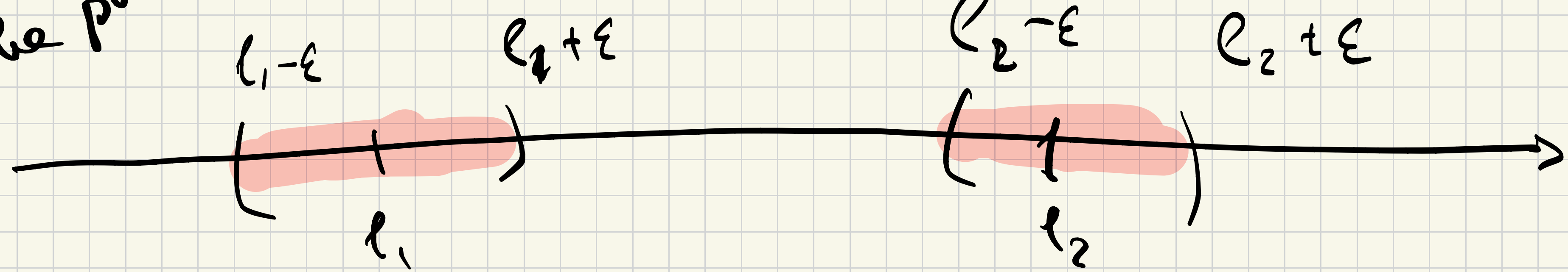
an so $|a^n| < \varepsilon$ by previous computation

∎

Proposition Limit of convergent sequences is unique.

If ℓ_1 and ℓ_2 are limits of (x_n) then $\ell_1 = \ell_2$

Idea of the proof: take ε small enough s.t.



Proposition A convergent sequence is bounded.

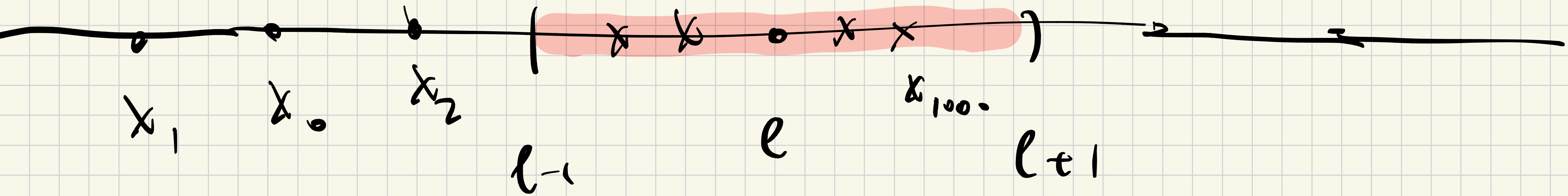
Proof sketch:

Let $\lim_{n \to \infty} x_n = \ell \in \mathbb{R}$

then for $\varepsilon = 1$ $\exists$ N s.t.

For all $n > N$ $|x_n - \ell| < 1$

$\Rightarrow$ For all $n > N$ $\ell - 1 < x_n < \ell + 1$



So the whole sequence is bounded

above by $\operatorname{Max}(x_0, x_1, x_2 \dots x_N, \ell+1)$ $+1$

below by $\operatorname{Min}(x_0, x_1, \dots, x_N, \ell-1)$ -1

to make inequalities strict

Definition: Divergent sequence

A sequence $(x_n)_{n \geq n_0}$ is divergient if it does not have a limit.

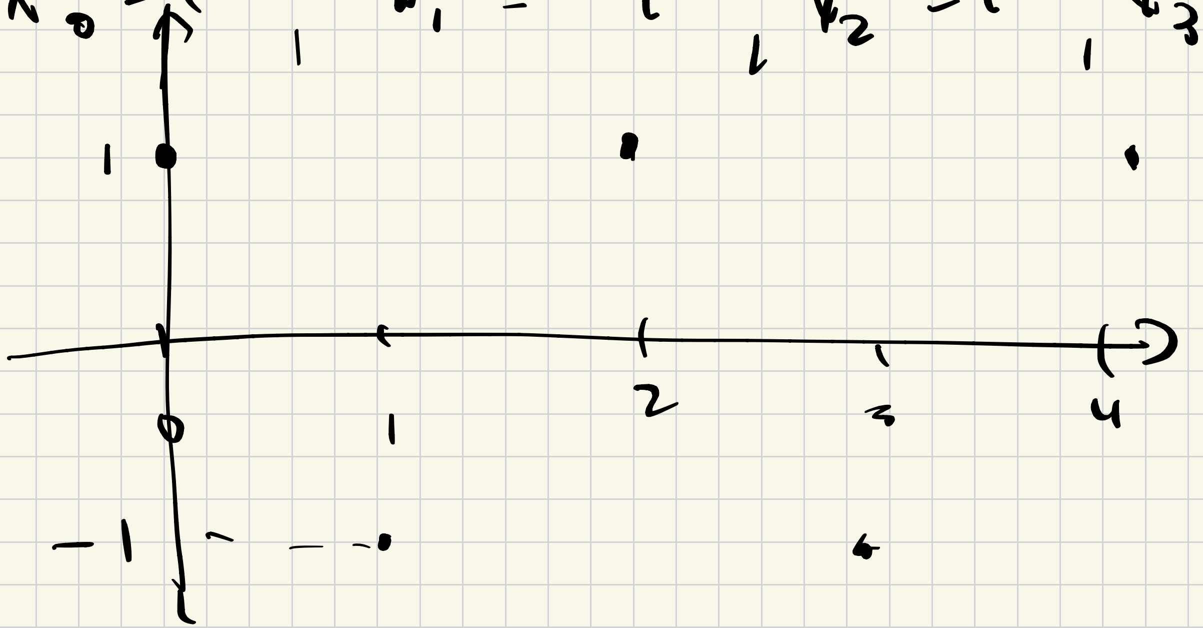
That is: Every $\ell \in \mathbb{R}$ is Not a limit of (x_n):

$\forall \ell \in \mathbb{R}$ " $\exists \varepsilon > 0$ s.t. $\forall N \in \mathbb{N}$

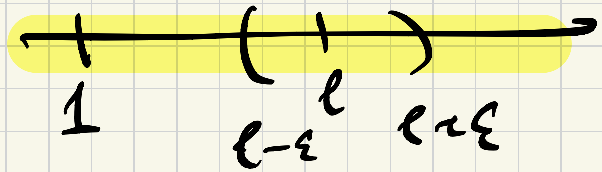
$\exists n > N$ s.t. $|x_n - \ell| \geq \varepsilon$ ".

Example $\quad x_n = (-1)^n$

$x_0 = 1, \quad x_1 = -1, \quad x_2 = 1, \quad x_3 = -1 \quad \dots$



$x_n = (-1)^n$ diverges



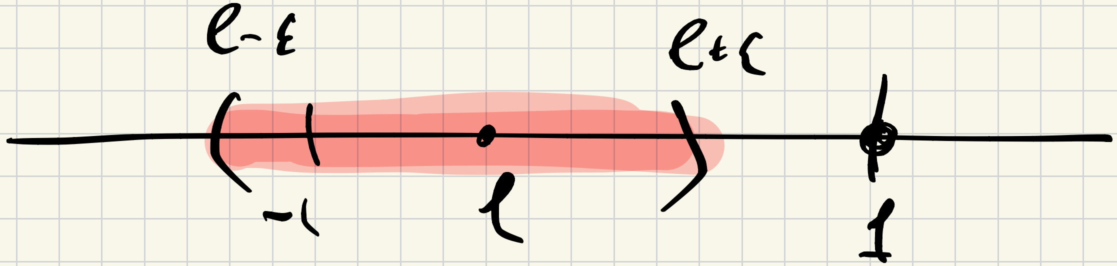
Proof: Need to show that $\forall \ell \in \mathbb{R}$

$\exists \varepsilon > 0$ s.t. $\forall N \in \mathbb{N}$ $\exists n > N$ with

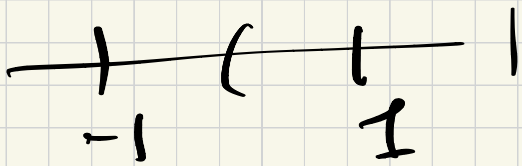
$$|x_n - \ell| > \varepsilon.$$

Take any $\ell \neq +1$ then for $\varepsilon = \frac{|\ell - 1|}{2}$

there is no N s.t. $\forall n > N$ $|x_n - \ell| < \varepsilon$



So $\forall N \in \mathbb{N}$ $\exists$ an even number $n > N$ therefore $x_n = 1$ is s.t. $|x_1 - \ell| > \varepsilon$.



Similar argument works for $\ell = 1$.

Indeed $\varepsilon = 1$ and then all odd members of the sequence are equal to -1 and outside of $(0, 2)$ $\blacksquare$

Example $x_n = \sqrt[3]{n}$

x_n does not converge since it is unbounded.

Limits algebra

Proposition Let (x_n), (y_n) be two convergent sequences with $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$ then

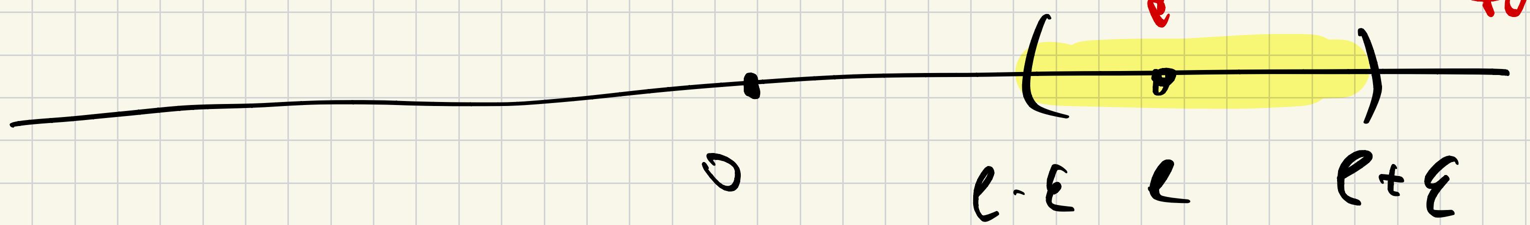
1. Sequence $(x_n + y_n)$ is convergient and $\lim_{n\to\infty} (x_n + y_n) = x + y$
2. Sequence $(x_n \cdot y_n)$ is convergient and $\lim_{n\to\infty} (x_n \cdot y_n) = x \cdot y$
3. If also $y \neq 0$, sequence $\left(\frac{x_n}{y_n}\right)$ is convergient and $\lim \left(\frac{x_n}{y_n}\right) = \frac{x}{y}$
4. If $\exists N \in \mathbb{N}$ s.t. $x_n \lesseqgtr y_n \quad \forall n > N$ then $x \lesseqgtr y$.

About part 3.

If $\lim_{n\to\infty} y_n \neq 0$ that means that

$y_n = 0$ only finitely many times.

Here starting from N



in particular $x_n \neq 0$ for $n > N$.

So it makes sense to talk about sequence $\frac{x_n}{y_n}$ by removing entries for which $y_n = 0$ (which are finitely many)

Example Let $\alpha, \beta \in \mathbb{R}$

and $\lim\limits_{n\to\infty} x_n = x$; $\lim\limits_{n\to\infty} y_n = y$ then

$$\lim_{n\to\infty} (\alpha x_n + \beta y_n) = \alpha x + \beta y$$